JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 30, No. 3, August 2017
http://dx.doi.org/10.14403/jcms.2017.30.3.341

A STUDY ON THE SET-VALUED DISCRETE DYNAMICAL SYSTEM $(2^X,\ \overline{f})$

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Abstract. This paper is devoted to some dynamical properties such as transitivity, mixing and specification of two discrete dynamical systems $(X,\ f)$ and $(2^X,\ \overline{f})$ on compact metric spaces.

1. Introduction and backgrounds

S. Li [4] proved that the shift map $\sigma_f : \varprojlim(X,\ f) \leftarrow \varprojlim(X,\ f)$ induced by a continuous map f on a compact metric space X is chaotic in the sense of Devaney if and only if (X, f) is chaotic in the sense of Devaney. Moreover, Roman-Flore [6] showed that the Devaney's chaoticity of (X, f) implies the Devaney's chaoticity of the set-valued dynamical system $(2^X, \overline{f})$, and gave a question whether the converse of the statement in true or not.

In this paper we give a partial answer about the above question, and study some relationship between two dynamics of (X, f) and $(2^X, \overline{f})$. More precisely, we show that if Devaney chaotic and weak mixing then $\overline{f}$ is Devaney chaotic; f is strong [resp. mild] mixing if and only if $\overline{f}$ is strong [resp. mild] mixing , respectively; f is has a specification [resp. Property P] if and only if $\overline{f}$ has specification. [Property P], respectively.

We start with the definition of Devaney chaos. Let $(X,\ d)$ be a compact metric space and $f : X \to X$ be a continuous map. A map f is called to be *Devaney chaotic* [2] if f satisfies the following three conditions.

(1) f is *transitive* that is for every pair $U,\ V$ of non-empty open subsets of X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

Received May 31, 2017; Accepted July 24, 2017.
2010 Mathematics Subject Classification: Primary 37B40.
Key words and phrases: set-valued discrete dynamical systems, Deveney's chaoticity, mixing, specification.

(2) f is *periodically dense* that is the set of periodic points of f is dense in X.
(3) f has *sensitive dependence on initial conditions* that is there exists a $\delta > 0$ such that for any $x \in X$, there exist a sequence (x_k) in X and a sequence (n_k) of positive integers such that $\lim_{k\to\infty} x_k = x$ and $d(f^{n_k}(x),\ f^{n_k}(x_k)) \geq \delta$ for all k.

2. Preliminaries

In this paper, we will investigate the relationships between the mixing property of $(2^X,\ \overline{f})$ and the mixing property of $(X,\ f)$. In addition, we discuss totally transitivity and specification for the set-valued discrete dynamical system $(2^X,\ \overline{f})$.

For a compact metric space $(X,\ d)$, let 2^X be the family of all non-empty compact subsets of X. A metric H on 2^X is defined as follows:

Definition 2.1. If $\epsilon > 0$ and $A \in 2^X$, then

$$N(A,\ \epsilon) = \{x \in X | d(x,\ a) < \epsilon \text{ for some } a \in A\}.$$

If $A,\ B \in 2^X$, then define

$$H(A,\ B) = \inf\{\epsilon > 0 | A \subset N(B,\ \epsilon) \text{ and } B \subset N(A,\ \epsilon)\}.$$

In fact, as will be proved in Theorem 2.2, H is a metric on 2^X. It will be called the *Hausdorff metric* induced by d.

Theorem 2.2. *The function* $H : 2^X \times 2^X \to [0,\ \infty)$ *is a metric on* 2^X.

Proof. We will prove the triangle inequality. Let $A,\ B,\ C \in 2^X$. We will show that

$$(*)\ H(A,\ C) \leq H(A,\ B) + H(B,\ C).$$

To prove (*), let $\eta > 0$ and let $\delta = \eta/2$. From the definition of H we see that

(1) $A \subset N(B,\ H(A,\ B) + \delta)$, and
(2) $B \subset N(C,\ H(B,\ C) + \delta)$.

Let $a \in A$. By (1), there exists $b \in B$ such that

(3) $d(a,\ b) < H(A,\ B) + \delta$.

By (2), there exists $c \in C$ such that

(4) $d(b,\ c) < H(B,\ C) + \delta$.

Using (3), (4), and the definition of δ, it follows easily that

$$d(a,\ c) < H(A,\ B) + H(B,\ C) + \eta.$$

Therefore, since a was an arbitrary point of A, we have proved that

(5) $A \subset N(C,\ H(A,\ B) + H(B,\ C) + \eta)$.

A similar argument shows that

(6) $C \subset N(A,\ H(A,\ B) + H(B,\ C) + \eta)$.

Since η was an arbitrary positive number, it follows from (5), (6), and the definition of $H(A,\ C)$ that (*) holds. This completes the proof of Theorem 2.2. □

Definition 2.3. Let (A_n) be a sequence in 2^X. Then define

$$\liminf A_n = \{x \in X |\ \text{if } U \text{ is a neighborhood of } x, \text{ then } U \cap A_n \neq \emptyset \text{ for all but finitely many } n\}$$

$$\limsup A_n = \{x \in X |\ \text{if } U \text{ is a neighborhood of } x, \text{ then } U \cap A_n \neq \emptyset \text{ for infinitely many } n\}$$

If $\liminf A_n = A = \limsup A_n$, then we say that the sequence (A_n) converges to A, written $\lim_{n\to\infty} A_n = A$.

Remark 2.4. Let (A_n) be a sequence in 2^X. As is easy to verify :

(1) $\liminf A_n \subset \limsup A_n$.
(2) $\liminf A_n$ and $\limsup A_n$ are each closed subsets of X.
(3) If (A_{n_i}) is a subsequence of (A_n), then
$\liminf A_n \subset \liminf A_{n_i}$ and $\limsup A_{n_i} \subset \limsup A_n$.

Theorem 2.5. *Let (A_n) be a sequence in 2^X. If (A_n) converges to A in the sense of Definition 2.3, then $A \in 2^X$ and (A_n) converges to A with respect to the Hausdorff metric. Conversely, if (A_n) converges with respect to the Hausdorff metric to A, then (A_n) converges to A in the sense of Definition 2.3.*

Proof. First assume (A_n) converges to A in the sense of Definition 2.3. Since X is compact and each $A_n \neq \emptyset$, it follows that $\limsup A_n \neq \emptyset$. Thus, since $A = \limsup A_n$, we have that $A \neq \emptyset$. Also, by Remark 2.4, A is a compact subset of X. Hence $A \in 2^X$. Now we show that (A_n) converges to A with respect to the Hausdorff metric. Let $\epsilon > 0$. Note that

(1) $\limsup A_n = A \subset N_d(A,\ \epsilon)$

and that, since $N_d(A,\ \epsilon)$ is an open subset of X,

(2) the complement of $N_d(A,\ \epsilon)$ is a compact subset of X.

Using (1), (2), and Remark 2.4, it follows that there exists a natural number N_1 such that

(3) $A_n \subset N_d(A,\ \epsilon)$ for each $n \geq N_1$.

Since A is a non-empty compact subset of X, there exist a finite number of open subsets $U_1, \cdots, U_k$ of X such that $A \subset \cup_{i=1}^{k} U_i$, the diameter of each U_i is less than ϵ, and $U_i \cap A \neq \emptyset$ for each $i = 1, \cdots, k$. Then, since $A = \liminf A_n$, there exists for each $i = 1, \cdots, k$ a natural number M_i such that $A_n \cap U_i \neq \emptyset$ whenever $n \geq M_i$. Let $N_2 = \max\{M_1, \cdots, M_k\}$. It is easy to verify that

(4) $A \subset N_d(A_n,\ \epsilon)$ for each $n \geq N_2$.

Let $N = \max\{N_1,\ N_2\}$. Then, by (3), (4), and the definition of H we see that $H(A,\ A_n) < \epsilon$ for each $n \geq N$. Therefore we have proved that (A_n) converges to A with respect to the Hausdorff metric. This proves half of Theorem. To prove the other half, assume (A_n) converges to an $A \in 2^X$ with respect to the Hausdorff metric. We first show that

(5) $\limsup A_n \subset A$.

To verify (5), let $\epsilon > 0$. Since (A_n) converges to A with respect to the Hausdorff metric, there exists a natural number N such that $H(A,\ A_n) < \epsilon$ for each $n \geq N$. This implies that no point of $\limsup A_n$ can be more that ϵ from every pont of A. Therefore, since $\epsilon > 0$ was arbitrary, we have proved (5). Next we show that

(6) $A \subset \liminf A_n$.

To verify (6), let $\epsilon > 0$. Let $a_0 \in A$ and let $U = N_d(a_0,\ \epsilon)$. Since (A_n) converges to A with respect to the Hausdorff metric, there exists a natural number N such that $H(A,\ A_n) < \epsilon$ for each $n \geq N$. Hence, by the definition of H, we have that $A \subset N_d(A_n,\ \epsilon)$ for each $n \geq N$. Thus $U \cap A_n \neq \emptyset$ for each $n \geq N$. Therefore, since $\epsilon > 0$ was arbitrary,

$$a_0 \in \liminf A_n.$$

This proves (6). Combining (5) and (6) and using Remark 2.4, we see that (A_n) converges to A in the sense of Definition 2.3. □

THEOREM 2.6. *The space 2^X is compact.*

Proof. To prove that 2^X is compact, it suffices by Theorem 2.5 to show that every sequence in 2^X has a convergent subsequence in the sense of Definition 2.3. To do this, let (A_n) be a sequence in 2^X. We define sequences

$$\begin{array}{l} (A_n^1) \ : \ A_1^1,\ A_2^1,\ \cdots,\ A_n^1,\ \cdots \\ (A_n^2) \ : \ A_1^2,\ A_2^2,\ \cdots,\ A_n^2,\ \cdots \end{array}$$

$$\vdots$$
$$(A_n^n) \ : \ A_1^n, \ A_2^n, \ \cdots, \ A_n^n, \ \cdots$$
$$\vdots$$

inductively as follows. Let $\beta = \{U_n\}$ be a countable basis for X. Define (A_n^1) by $A_n^1 = A_n$ for each $n = 1, \ 2, \ \cdots$. Assume inductively that we have defined the sequence (A_n^k). We define (A_n^{k+1}) in one of the following two ways :

(1) If (A_n^k) has a subsequence $(A_{n_i}^k)$ such that $(\limsup A_{n_i}^k) \cap U_k = \emptyset$, then let (A_n^{k+1}) be one such subsequence of (A_n^k).
(2) If every subsequence of (A_n^k) has a point of its lim sup in U_k, then let (A_n^{k+1}) be given by $A_n^{k+1} = A_n^k$ for each $n = 1, \ 2, \ \cdots$.

Now, having defined the sequence (A_n^k) for each $k = 1, \ 2, \ \cdots$, consider the 'diagonal sequence' (A_n^n). Clearly (A_n^n) is a subsequence of (A_n), and we will show it converges. Suppose (A_n^n) does not converges. Then, by Remark 2.4, there exists a point $p \in \limsup A_n^n$ such that $p \notin \liminf A_n^n$. Hence, there exists $U_m \in \beta$ such that $p \in U_m$ and such that $U_m \cap A_{n_i}^{n_i} = \emptyset$ for some subsequence $(A_{n_i}^{n_i})$ of (A_n^n). Clearly $(A_{n_i}^{n_i})$ is a subsequence of (A_n^m). Thus (A_n^m) satisfies (1) above. Hence

$$(\limsup A_n^{m+1}) \cap U_m = \emptyset.$$

Therefore, since $(A_n^n)_{n=m+1}^{\infty}$ is a subsequence of (A_n^{m+1}), it follows using Remark 2.4 that $(\limsup A_n^n) \cap U_m = \emptyset$. But, since $p \in (\limsup A_n^n) \cap U_m$, we have a contradiction. Therefore (A_n^n) converges. This completes the proof of Theorem. □

For any finite non-empty open subsets $U_1, \ \cdots, \ U_n$ of X, let

$$< U_1, \ \cdots, \ U_n >= \{A \in 2^X \mid A \subset \cup_{i=1}^n U_i \text{ and } A \cap U_i \neq \emptyset \text{ for all } 1 \leq i \leq n\}.$$

THEOREM 2.7. *Let $(X, \ d)$ be a compact metric space. Then the set β of all subsets of 2^X of the form $< U_1, \ \cdots, \ U_n >$ is a basis for 2^X.*

Proof. First we show that $< U_1, \ \cdots, \ U_n >$ is an open subset of 2^X. Let

$$A \in < U_1, \ \cdots, \ U_n > .$$

Then $A \subset \cup_{i=1}^n U_i$. For each $i = 1, \ \cdots, \ n$, since $A \cap U_i \neq \emptyset$, we can choose a point $a_i \in A \cap U_i$. There exists $\epsilon > 0$ such that

$$N_d(A, \ \epsilon) \subset \cup_{i=1}^n U_i \text{ and } N_d(a_i, \ \epsilon) \subset U_i \text{ for all } 1 \leq i \leq n.$$

Let $B \in N_H(A,\ \epsilon)$. Then $B \subset N_d(A,\ \epsilon) \subset \cup_{i=1}^{n} U_i$. For each $i = 1,\ \cdots,\ n$, since

$$a_i \in A \subset N_d(B,\ \epsilon),$$

there is a point $b_i \in B$ such that $d(a_i,\ b_i) < \epsilon$. Since $b_i \in N_d(a_i,\ \epsilon) \subset U_i$, we have

$$B \cap U_i \neq \emptyset \text{ for all } 1 \leq i \leq n.$$

Thus $B \in < U_1,\ \cdots,\ U_n >$ so $N_H(A,\ \epsilon) \subset < U_1,\ \cdots,\ U_n >$. Hence $< U_1,\ \cdots,\ U_n >$is an open subset of 2^X.

Next we show that β is a basis for 2^X. Let α be an open subset of 2^X. Given any $A \in \alpha$, there exists $\epsilon > 0$ such that $N_H(A,\ \epsilon) \subset \alpha$. Since A is compact, there exist finitely many points $a_1,\ \cdots,\ a_n$ of A such that

$$A \subset \cup_{i=1}^{n} N_d\Big(a_i,\ \frac{\epsilon}{3}\Big).$$

Clearly, $A \in < N_d(a_1,\ \frac{\epsilon}{3}),\ \cdots,\ N_d(a_n,\ \frac{\epsilon}{3}) >$. Let $B \in < N_d(a_1,\ \frac{\epsilon}{3}),\ \cdots,\ N_d(a_n,\ \frac{\epsilon}{3}) >$. For any $b \in B$, since $B \subset \cup_{i=1}^{n} N_d(a_i,\ \frac{\epsilon}{3})$, there exists i such that $b \in N_d(a_i,\ \frac{\epsilon}{3})$. Thus

$$B \subset N_d\Big(A,\ \frac{\epsilon}{3}\Big).$$

For any $a \in A$, there exists i such that $a \in N_d(a_i,\ \frac{\epsilon}{3})$, that is, $d(a_i,\ a) < \frac{\epsilon}{3}$. Since

$$B \cap N_d\Big(a_i,\ \frac{\epsilon}{3}\Big) \neq \emptyset,$$

we can choose a point $b \in B$ such that $d(a_i,\ b) < \frac{\epsilon}{3}$. Then we have

$$d(a,\ b) \leq d(a,\ a_i) + d(a_i,\ b) < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon.$$

Thus $A \subset N_d(B,\ \frac{2}{3}\epsilon)$. Hence $H(A,\ B) \leq \frac{2}{3}\epsilon < \epsilon$ so we get

$$A \in < N_d\Big(a_1,\ \frac{\epsilon}{3}\Big),\ \cdots,\ N_d\Big(a_n,\ \frac{\epsilon}{3}\Big) > \subset N_H(A,\ \epsilon) \subset \alpha.$$

Therefore β is a basis for 2^X. □

If $f : X \to X$ is a continuous map then by $\overline{f}(A) = \{f(a) | a \in A\}$ for every $A \in 2^X$ one defines a continuous map $\overline{f} : 2^X \to 2^X$.

A map $f : X \to X$ is called to be *totally transitive* if $f^n : X \to X$ is transitive for every positive integer n. A map $f : X \to X$ is called to be *weak mixing* if the product map $f \times f : X \times X \to X \times X$ is transitive, and f is *strong mixing* if for any two non-empty open subsets $U,\ V$ of X there is a positive integer N such that

$$f^n(U) \cap V \neq \emptyset$$

for every integer $n \geq N$.

In [8], Xiong and Yang investigated the chaos caused by a mixing map and revealed a kind of quite complex phenomenon.

A map $f : X \to X$ is called to be *weakly chaotic* in the sense of Xiong if there is a $c-$dense $F_\sigma-$-subset C of X such that for any subset A of C and any continuous map $F : A \to X$ there is an increasing positive integer sequence (p_n) such that

$$\lim_{n\to\infty} f^{p_n}(x) = F(x)$$

for each $x \in A$. f is called to be *chaotic* in the sense of Xiong if for any given increasing positive integer sequence (p_n) there is a $c-$dense F_σ-subset C of X such that for any subset A of C and any continuous map $F : A \to X$ there is a subsequence (p_{n_i}) of (p_n) such that $\lim_{i\to\infty} f^{p_{n_i}}(x) = F(x)$ for each $x \in A$.

3. Mixing and transivity

Let A be a subset of X. Then we define the extension of A to 2^X as

$$e(A) = \{K \in 2^X | K \subset A\}.$$

The following lemma is cited from [6].

LEMMA 3.1. *If U is a non-empty open subset of X, then*

(1) *$e(U) \neq \emptyset$ if and only if $U \neq \emptyset$.*
(2) *$e(U)$ is a non-empty open subset of 2^X.*
(3) *$e(U \cap V) = e(U) \cap e(V)$.*
(4) *$\overline{f}^n = \overline{f^n}$ for all positive integer n.*
(5) *$\overline{f}(e(U)) \subset e(\overline{f}(U))$.*

In [7], Shao studied the double properties and family versions of mixing.

THEOREM 3.2. [7] *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then the following conditions are equivalent.*

(1) *f is weak mixing.*
(2) *$\overline{f}$ is weak mixing.*
(3) *$\overline{f}$ is transitive.*

REMARK 3.3. [6] Roman-Flores showed that $\overline{f}$ transitive implies f transitive. Hence, Theorem 3.2 generalizes the result of Roman-Flores. In addition, the theorem shows that the concepts of weak mixing and transitivity coincide for the set-valued discrete dynamical system $(2^X, \overline{f})$.

Devaney's chaoticity of f does not imply Devaney's chaoticity of $\overline{f}$ as shown by the following example.

EXAMPLE 3.4. *Let $X = \{0,\ 1\}$ be a discrete space and $f : X \to X$ be a continuous map with $f(0) = 1$ and $f(1) = 0$. Clearly, f is transitive and periodically dense. so f is Devaney chaotic. Since $2^X = \{\{0\},\ \{1\},\ \{0,\ 1\}\}$ is a discrete space with two periodic orbits, $\overline{f}$ is not Devaney chaotic.*

However, we have the following result.

THEOREM 3.5. *Let X be a compact metric space and $f : X \to X$ be a continuous map. If f is Devaney chaotic and weak mixing, then $\overline{f}$ is Devaney chaotic.*

Proof. By Theorem 3.2, it is necessary to prove that if $Per(f)$ is dense in X then $Per(\overline{f})$ is dense in 2^X, where $Per(f)$ is the set of periodic points of f.

In fact, let α be a non-empty open subset of 2^X. We choose non-empty open subsets $U_1,\ \cdots,\ U_k$ of X such that

$$< U_1,\ \cdots,\ U_k > \subset \alpha.$$

Since $Per(f)$ is dense in X, there are periodic points $p_1,\ \cdots,\ p_k$ of f such that $p_i \in U$ for every $i = 1,\ \cdots,\ k$. Let $A = \{p_1,\ \cdots,\ p_k\}$. Then $A \in Per(\overline{f})$ and $A \in < U_1,\ \cdots,\ U_k > \subset \alpha$.

This shows that $Per(\overline{f})$ is dense in 2^X. □

THEOREM 3.6. *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then the following conditions are equivalent.*

(1) *$\overline{f}$ is strong mixing.*

(2) *f is strong mixing.*

Proof. (1)⇒(2): Suppose that $\overline{f}$ is strong mixing. Let U, V be any two non-empty open subsets of X. Since $e(U)$, $e(V)$ are non-empty open subsets of 2^X, there is a positive integer N such that

$$\overline{f}^n(e(U)) \cap e(V) \neq \emptyset \text{ for all } n \geq N.$$

By Lemma 3.1, we have

$$\overline{f}(e(U)) \cap e(V) \subset e(f^n(U)) \cap e(V) = e(f^n(U) \cap V).$$

Thus

$$f^n(U) \cap V \neq \emptyset \text{ for all } n \geq N.$$

This shows that f is strong mixing.

(2)⇒(1): Suppose that $\overline{f}$ is strong mixing. Let α and β be any two non-empty open subsets of 2^X. We choose non-empty open subsets

$$U_1, \ \cdots, \ U_k, \ V_1, \ \cdots, \ V_k$$

of X such that

$$< U_1, \ \cdots, \ U_k > \subset \alpha \text{ and } < V_1, \ \cdots, \ V_k > \subset \beta.$$

For each $i = 1, \ \cdots, \ k$, since f is strong mixing, there exists a positive integer N_i such that

$$f^n(U_i) \cap V_i \neq \emptyset \text{ for all } n \geq N_i.$$

Let $N = \max\{N_1, \ \cdots, \ N_k\}$. For every integer $n \geq N$, since $f^n(U_i) \cap V_i \neq \emptyset$, there is a point $x_i \in U_i$ such that $f^n(x_i) \in V_i$. Let $A = \{x_1, \ \cdots, \ x_k\}$. Then $A \in 2^X$ and

$$A \in < U_1, \ \cdots, \ U_k > \subset \alpha \text{ and } \overline{f}^n(A) \in < V_1, \ \cdots, \ V_k > \subset \beta.$$

Thus $\overline{f}^n(\alpha) \cap \beta \neq \emptyset$ for all $n \geq N$. This shows that $\overline{f}$ is strong mixing. □

THEOREM 3.7. *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then the following conditions are equivalent.*

(1) *$\overline{f}$ is mild mixing.*
(2) *f is mild mixing.*

Proof. (1)⇒(2): Suppose that $\overline{f}$ is mild mixing. For any transitive dynamical system $(Y, \ g)$, we will show that $(X \times Y, \ f \times g)$ is a transitive dynamical system.

Let $U_1 \times V_1$ and $U_2 \times V_2$ be two non-empty open subsets of $X \times Y$. Then $e(U_1) \times V_1$ and $e(U_2) \times V_2$ are two non-empty open subsets of $2^X \times Y$. Since $\overline{f} \times g$ is transitive, there are a positive integer n and a point $(A, \ y) \in e(U_1) \times V_1$ such that

$$(\overline{f} \times g)^n(A, \ y) = (\overline{f}^n(A), g^n(y)) \in e(U_2) \times V_2.$$

We pick a point $x \in A \subset U_1$. Then $f^n(x) \in \overline{f}^n(A) \subset U_2$. Thus $(x, \ y) \in U_1 \times V_1$ and

$$(f \times g)^n(x, \ y) = (f^n(x), g^n(y)) \in U_2 \times V_2.$$

Hence $(X \times Y, \ f \times g)$ is a transitive dynamical system. This shows that f is mild mixing.

(2)⇒(1): Suppose that f is mild mixing. For any transitive dynamical system $(Y,\ g)$, we will show that $(2^X \times Y,\ \overline{f} \times g)$ is a transitive dynamical system.

Let $\alpha_1 \times V_1$ and $\alpha_2 \times V_2$ be two non-empty open subsets of $2^X \times Y$. Then there are non-empty open subsets $U_1^1,\ \cdots,\ U_k^1,\ U_1^2,\ \cdots,\ U_k^2$ of X such that

$$< U_1^1,\ \cdots,\ U_k^1 > \subset \alpha_1 \text{ and } < U_1^2,\ \cdots,\ U_k^2 > \subset \alpha_2.$$

Since f is mild mixing, $(X \times Y,\ f \times g)$ is a transitive dynamical system. By using the induction, we get that

$$(X^k \times Y,\ f \times \cdots \times f \times g)$$

is a transitive dynamical system. Thus, for two non-empty open subsets

$$U_1^1 \times \cdots \times U_k^1 \times V_1 \text{ and } U_1^2 \times \cdots \times U_k^2 \times V_2$$

of $X^k \times Y$, there are a positive integer n and a point

$$(x_1,\ \cdots,\ x_k,\ y) \in U_1^1 \times \cdots \times U_k^1 \times V_1$$

such that

$$\begin{aligned}(f \times \cdots \times f \times g)^n(x_1,\ \cdots,\ x_k,\ y) &= (f^n(x_1),\ \cdots,\ f^n(x_k),\ g^n(y)) \\ &\in U_1^2 \times \cdots \times U_k^2 \times V_2.\end{aligned}$$

Thus $f^n(x_i) \in U_i^2$ for every $i = 1,\ \cdots,\ k$ and $g^n(y) \in V_2$. Let $A = \{x_1,\ \cdots,\ x_k\}$. Then

$$A \in < U_1^1,\ \cdots,\ U_k^1 > \subset \alpha_1 \text{ and } \overline{f}^n(A) \in < U_1^2,\ \cdots,\ U_k^2 > \subset \alpha_2.$$

Hence

$$(A,\ y) \in \alpha_1 \times V_1 \text{ and } (\overline{f}^n(A),\ g^n(y)) = (\overline{f} \times g)^n(A,\ y) \in \alpha_2 \times V_2.$$

Therefore $(2^X \times Y,\ \overline{f} \times g)$ is a transitive dynamical system. This shows that $\overline{f}$ is mild mixing. $\square$

4. Specification and totally transitivity

Definition 4.1. A map $f : X \to X$ is called to have *specification* if for any positive number ϵ there is a positive number $M(\epsilon)$ such that for any integer $k \geq 2$ and any k points $x_1, \cdots, x_k$ of X, and any $2k$ non-negative integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$$

with $a_i - b_{i-1} \geq M(\epsilon)$ for each $i = 2, \cdots, k$ there is a point x of X satisfying

$$d(f^n(x), f^n(x_i)) < \epsilon$$

for every $n = a_i, \cdots, b_i$ and every $i = 2, \cdots, k$.

Theorem 4.2. *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then the following conditions are equivalent.*

(1) *$\overline{f}$ has specification.*
(2) *f has specification.*

Proof. (1)⇒(2): Suppose that $\overline{f}$ has specification. Let $\epsilon > 0$ and let $M = M(\epsilon)$ be a positive number as in the definition of specification for $\overline{f}$. For any integer $k \geq 2$, we take any k points $x_1, \cdots, x_k$ of X and any $2k$ non-negative integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$$

with $a_i - b_{i-1} \geq M$ for every $i = 2, \cdots, k$. We denote $A_i = \{x_i\}$ for each $i = 1, \cdots, k$, then $A_1, \cdots, A_k \in 2^X$. Since $\overline{f}$ has specification, there is a point A of 2^X such that

$$H(\overline{f}^n(A), \overline{f}^n(A_i)) < \epsilon$$

for all $n = a_i, \cdots, b_i$ and all $i = 1, \cdots, k$. Since

$$\overline{f}^n(A) \subset N(\overline{f}^n(A_i), \epsilon) = N(f^n(x_i), \epsilon),$$

we pick a point $x \in A$, then we have

$$d(f^n(x), f^n(x_i)) < \epsilon$$

for all $n = a_i, \cdots, b_i$ and all $i = 1, \cdots, k$. Thus f has specification.

(2)⇒(1): Suppose that f has specification. Let $\epsilon > 0$ and let $M = M(\frac{\epsilon}{3})$ be a positive number as in the definition of specification for f. For any integer $k \geq 2$, we take any k points $A_1, \cdots, A_k$ of 2^X and any $2k$ non-negative integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$$

with $a_i - b_{i-1} \geq M$ for every $i = 2, \cdots, k$. Since $f, \cdots, f^{b_k}$ are uniformly continuous, there exists a $\delta > 0$ such that

$$d(x, y) < \delta \text{ implies } d(f^i(x), f^i(y)) < \tfrac{\epsilon}{3}$$

for all $i = 0, 1, \cdots, b_k$. For each $i = 1, \cdots, k$, $\{N(x, \delta) | x \in A_i\}$ is an open cover of A_i. Since A_i is compact, there are finitely many points $x^i_1, \cdots, x^i_{m_i}$ of A_i such that

$$A_i \subset \cup_{t=1}^{m_i} N(x^i_t, \delta).$$

For k points $x^1_{t_1}, \cdots, x^k_{t_k}$ of X where $1 \leq t_i \leq m_i$ and $1 \leq i \leq k$, since f has specification, there is a point $x(t_1, \cdots, t_k)$ of X such that

$$d(f^n(x(t_1, \cdots, t_k)), f^n(x^i_{t_i})) < \frac{\epsilon}{3}$$

for all $n = a_i, \cdots, b_i$ and all $i = 1, \cdots, k$. Let

$$A = \{x(t_1, \cdots, t_k\} | 1 \leq i \leq k, 1 \leq t_i \leq m_i\}.$$

Then $A \in 2^X$. Let $i = 1, \cdots, k$ and $1 \leq t_i \leq m_i$. Since

$$d(f^n(x(t_1, \cdots, t_k)), \overline{f}^n(A_i)) \leq d(f^n(x(t_1, \cdots, t_k)), f^n(x^i_{t_i})) < \frac{\epsilon}{3},$$

we have

$$\overline{f}^n(A) \subset N\Big(\overline{f}^n(A_i), \frac{\epsilon}{3}\Big)$$

for all $n = a_i, \cdots, b_i$. Given any $x \in A_i$, there exists a t_i such that $x \in N(x^i_{t_i}, \delta)$. Since $d(x^i_{t_i}, x) < \delta$, we have $d(f^n(x^i_{t_i}), f^n(x)) < \frac{\epsilon}{3}$ for all $n = a_i, \cdots, b_i$. Choose any

$$t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k$$

then we have

$$\begin{aligned} d(f^n(x), \overline{f}^n(A)) &\leq d(f^n(x), f^n(x(t_1, \cdots, t_k)) \\ &\leq d(f^n(x), f^n(x^i_{t_i})) + d(f^n(x^i_{t_i}), f^n(x(t_1, \cdots, t_k)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon \end{aligned}$$

Thus $\overline{f}^n(A_i) \subset N(\overline{f}^n(A), \frac{2}{3}\epsilon$. Hence

$$H(\overline{f}^n(A), \overline{f}^n(A_i)) \leq \frac{2}{3}\epsilon < \epsilon$$

for all $i = a_i, \cdots, b_i$. Therefore $\overline{f}$ has specification. □

In [1], Bowen introduced the concept of Property P to characterize chaotic phenomenon of flow with the specification property.

A map $f : X \to X$ is called to have *Property* P if for any two non-empty open subsets $U_1,\ U_2$ of X there exists a positive integer N such that, for any integer $k \geq 2$ and any $s = (s(1),\ s(2),\ \cdots,\ s(k)) \in \{1,\ 2\}^k$ there is a point x of X satisfying

$$x \in U_{s(1)},\ f^N(x) \in U_{s(2)},\ \cdots,\ f^{(k-1)N}(x) \in U_{s(k)}.$$

THEOREM 4.3. [9] *Let X be a compact metric space and $f : X \to X$ be a continuous map. If f has Property P then f is weak mixing.*

THEOREM 4.4. *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then the following conditions are equivalent.*

(1) *$\overline{f}$ has Property P.*
(2) *f has Property P.*

Proof. (1)⇒(2): Suppose that $\overline{f}$ has Property P. Let U_1 and U_2 be any two non-empty open subsets of X. By Lemma 3.1, $e(U_1)$ and $e(U_2)$ are non-empty open subsets of 2^X. Since $\overline{f}$ has Property P, there is a positive integer N such that, for any integer $k \geq 2$ and any $s = (s(1),\ s(2),\ \cdots,\ s(k)) \in \{1,\ 2\}^k$, there is a point A of 2^X satisfying

$$A \in e(U_{s(1)}),\ \overline{f}^N(A) \in e(U_{s(2)}),\ \cdots,\ \overline{f}^{(k-1)N}(A) \in e(U_{s(k)}).$$

Thus we have

$$A \subset U_{s(1)},\ \overline{f}^N(A) \in U_{s(2)},\ \cdots,\ \overline{f}^{(k-1)N}(A) \subset U_{s(k)}.$$

Picking a point $x \in A$, then we have

$$x \in U_{s(1)},\ f^N(x) \in U_{s(2)},\ \cdots,\ f^{(k-1)N}(x) \in U_{s(k)}.$$

Hence f has Property P.

(2)⇒(1): Suppose that f has Property P. Let $\alpha_1,\ \alpha_2$ be any two non-empty open subsets of 2^X. Choose open subsets $U_1^1,\ \cdots,\ U_m^1,\ U_1^2,\ \cdots,\ U_m^2$ of X such that

$$< U_1^1,\ \cdots,\ U_m^1 >\subset \alpha_1 \text{ and } < U_1^2,\ \cdots,\ U_m^2 >\subset \alpha_2.$$

Since f has Property P, for two non-empty open subsets U_i^1 and U_i^2 of X there is a positive integer N_i such that for any integer $k_i \geq 2$ and any

$$s = (s(1),\ s(2),\ \cdots,\ s(k_i)) \in \{1,\ 2\}^{k_i},$$

there is a point x_i of X satisfying

$$x_i \in U_i^{s(1)},\ f^{N_i}(x_i) \in U_i^{s(2)},\ \cdots,\ f^{(k_i-1)N_i}(x_i) \in U_i^{s(k_i)}.$$

Let N denote the least common multiple of $N_1,\ N_2,\ \cdots,\ N_m$. For any integer $k \geq 2$ and any $s = (s(1),\ s(2),\ \cdots,\ s(k)) \in \{1,\ 2\}^k$, we denote

$$p_i = \frac{N}{N_i},\ i = 1,\ 2,\ \cdots,\ m$$

and

$$s = (s(1),\ \cdots,\ s(1),\ s(2),\ \cdots,\ s(2),\ \cdots,\ s(k),\ \cdots,\ s(k)) \in \{1,\ 2\}^{kp_i}.$$

Then there is a point y_i of X such that

$$\begin{gathered} y_i \in U_i^{s(1)},\ f^{N_i}(y_i) \in U_i^{s(1)},\ \cdots,\ f^{(p_i-1)N_i}(y_i) \in U_i^{s(1)} \\ f^{p_iN_i}(y_i) \in U_i^{s(2)},\ f^{(p_i+1)N_i}(y_i) \in U_i^{s(2)},\ \cdots,\ f^{(2p_i-1)N_i}(y_i) \in U_i^{s(2)} \\ \vdots \\ f^{(k-1)p_iN_i}(y_i) \in U_i^{s(k)}, f^{((k-1)p_i+1)N_i}(y_i) \in U_i^{s(k)}, \cdots, \\ f^{(kp_i-1)N_i}(y_i) \in U_i^{s(k)}. \end{gathered}$$

Thus we have

$$y_i \in U_i^{s(1)},\ f^N(y_i) \in U_i^{s(2)},\ \cdots,\ f^{(k-1)N}(y_i) \in U_i^{s(k)}.$$

Let $A = \{y_1,\ y_2,\ \cdots,\ y_m\}$. Then $A \in 2^X$ and

$$\begin{gathered} A \in < U_1^{s(1)},\ U_2^{s(1)},\ \cdots,\ U_m^{s(1)} > \subset \alpha_{s(1)} \\ \overline{f}^N(A) \in < U_1^{s(2)},\ U_2^{s(2)},\ \cdots,\ U_m^{s(2)} > \subset \alpha_{s(2)} \\ \vdots \\ \overline{f}^{(k-1)N}(A) \in < U_1^{s(k)},\ U_2^{s(k)},\ \cdots,\ U_m^{s(k)} > \subset \alpha_{s(k)}. \end{gathered}$$

Hence $\overline{f}$ has Property P. □

Finally, we discuss totally transitivity.

THEOREM 4.5. *Let X be a compact metric space and $f : X \to X$ be a continuous map. If $\overline{f}$ is totally transitive, then so is f. However, the converse is not true.*

Proof. Since $\overline{f}$ is totally transitive, for every positive integer k, $\overline{f}^k = \overline{f^k}$ is transitive. By Theorem 3.6 of [6], f^k is transitive. Thus f is totally transitive. □

In general, the converse of Theorem 4.5 is not true.

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